11 Holomorphy

11.1 Non-renormalization of the superpotential

Couplings in the superpotential can be regarded as background fields. If we integrate out physics above a scale μ (i.e. calculate the Wilsonian effective action) then the effective superpotential must be a holomorphic function of the couplings. Consider a theory renormalized at some scale Λ with a superpotential:

$$W_{\text{tree}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3. \tag{11.1}$$

Recall that since the R charge doesn't commute with the SUSY generator:

$$[R, Q_\alpha] = -Q_\alpha, \tag{11.2}$$

we have $R[\psi] = R[\phi] - 1$, $R[\theta] = 1$. Since

$$\mathcal{L} \supset \lambda\phi\psi\psi \tag{11.3}$$

must have zero R charge, so

$$3R[\phi] - 2 = 0 \tag{11.4}$$

So $R[W] = 2$. Alternatively we could get the same result by noting:

$$\mathcal{L}_{\text{int}} = \int d^2\theta W \tag{11.5}$$

Chiral supermultiplets are labeled by the R charge of the scalar component.

$$\begin{array}{cccc} & U(1) & \times & U(1)_R \\ \phi & 1 & & 1 \\ m & -2 & & 0 \\ \lambda & -3 & & -1 \end{array} \tag{11.6}$$

Non-zero values for m and λ explicitly break both $U(1)$ symmetries, but they still lead to selection rules.

The symmetries and holomorphy of the effective superpotential restrict it to be of the form

$$\begin{aligned} W_{\text{eff}} &= f(\phi, m, \lambda) && (11.7)\\ &= m\phi^2 h\left(\frac{\lambda\phi}{m}\right) && (11.8)\\ &= \sum_n a_n \lambda^n m^{1-n} \phi^{n+2} && (11.9) \end{aligned}$$

The limit $\lambda \to 0$ restricts $n \geq 0$, and the $m \to 0$ restricts $n \leq 1$ so

$$W_{\text{eff}} = \frac{m}{2}\phi^2 + g\phi^3 = W_{\text{tree}} \tag{11.10}$$

i.e. the superpotential is not renormalized.

11.2 Wavefunction Renormalization

$$\mathcal{L} = Z\partial_\mu \phi^* \partial^\mu \phi + iZ\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi \tag{11.11}$$

where

$$Z = Z(m, \lambda, m^\dagger, \lambda^\dagger, \mu, \Lambda) \tag{11.12}$$

If we integrate out modes down to $\mu > m$ we have

$$Z = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{\mu^2}\right) \tag{11.13}$$

If we integrate out modes down to scales below m we have

$$Z = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{mm^\dagger}\right) \tag{11.14}$$

So there is wavefunction renormalization, and the couplings of canonically normalized fields run. In our example the running couplings are

$$\frac{m}{Z}, \frac{\lambda}{Z^{\frac{3}{2}}} \tag{11.15}$$

11.3 Integrating Out

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H \phi^2 \tag{11.16}$$

This model has three global $U(1)$ symmetries:

$$\begin{array}{cccc} & U(1)_A & U(1)_B & U(1)_R \\ \phi_H & 1 & 0 & 1 \\ \phi & 0 & 1 & \frac{1}{2} \\ M & -2 & 0 & 0 \\ \lambda & -1 & -2 & 0 \end{array} \tag{11.17}$$

If we want to integrate out down to $\mu < M$, we can integrate out ϕ_H. The an term in the effective superpotential has the form

$$\phi^j M^k \lambda^p \tag{11.18}$$

To preserve the symmetries we must have $j = 4$, $p = 2$, and $k = -1$. By comparing with perturbation theory we find:

$$W_{\text{eff}} = -\frac{\lambda^2 \phi^4}{8M} \tag{11.19}$$

We could also derive this exact result using the algebraic equation of motion

$$\frac{\partial W}{\phi_H} \tag{11.20}$$

Another interesting example is

$$W = \frac{1}{2} M \phi_H^2 + \frac{\lambda}{2} \phi_H \phi^2 + \frac{y}{6} \phi_H^3 \tag{11.21}$$

Integrating out ϕ_H yields

$$W_{\text{eff}} = \frac{m^3}{3y^2} \left(1 - \frac{3\lambda y \phi^2}{2M^2} \mp \left(1 - \frac{\lambda y \phi^2}{M^2} \right) \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right) \tag{11.22}$$

The singularities in W_{eff} indicate points where ϕ_H becomes massless and we shouldn't have integrated it out.

11.4 The Holomorphic Gauge Coupling

Using $y^\mu \equiv x^\mu - i\theta\sigma^\mu\theta$ we can write a the gauge multiplet as a chiral superfield

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y) \tag{11.23}$$

Using

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2} \tag{11.24}$$

we can write the SUSY Yang-Mills Lagrangian as a superpotential term

$$\frac{1}{16\pi i} \int d^4x \int d^2\theta \, \tau W_\alpha^a W_\alpha^a + h.c. \tag{11.25}$$
$$= \int d^4x - \frac{1}{4g^2} F^{a\mu\nu} F_{\mu\nu}^a - \frac{\theta_{\text{YM}}}{32\pi^2} F^{a\mu\nu} \widetilde{F}_{\mu\nu}^a + \frac{i}{g^2} \lambda^{a\dagger} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a$$

where

$$\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta} \tag{11.26}$$

Note g only appears as a holomorphic parameter, but the gauge fields are not canonically normalized. Recall that the $F^{a\mu\nu}\widetilde{F}^a_{\mu\nu}$ term calculates the topological winding number of instanton gauge configurations, though it has no effect in pertubations theory. One instanton effects are suppressed by

$$e^{-S_{\rm int}} = e^{\frac{-8\pi^2}{g^2}} \tag{11.27}$$

Recall

$$\mu\frac{dg}{d\mu} = -\frac{bg^3}{16\pi^2} \tag{11.28}$$

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\ln\left(\frac{\Lambda}{\mu}\right) \tag{11.29}$$

If we integrate down to μ

$$W_{\rm eff} = \frac{\tau(\Lambda;\mu)}{16\pi i}W^a_\alpha W^a_\alpha \tag{11.30}$$

Since

$$\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi \tag{11.31}$$

is a symmetry

$$\tau = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) + \sum_{n=1}^{\infty}\left(\frac{\Lambda}{\mu}\right)^{bn} a_n \tag{11.32}$$

So the holomorphic gauge coupling only receives one-loop corrections and non-perturbative corrections.

References

[1] K. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," Nucl. Phys. Proc. Suppl. **45BC** (1996) 1,cd hep-th/9509066.